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UNIVERSAL SEQUENCES FOR COMPLETE GRAPHS

N. ALON
Bell Communications Research, Morristown, NJ 07960, USA; Department of Mathematics, Sackler Faculty of Exact Sciences, Tel Aviv University, Ramat Aviv, Tel Aviv, Israel

Y. AZAR and Y. RAVID
Department of Computer Science, Sackler Faculty of Exact Sciences, Tel Aviv University, Ramat Aviv, Tel Aviv, Israel

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An n-labeled complete digraph G is a complete digraph with $n+1$ vertices and $n(n+1)$ edges labeled $\{1,2,\dots,n\}$ such that there is a unique edge of each label emanating from each vertex. A sequence S in $\{1,2,\dots,n\}^*$ and a starting vertex of G define a unique walk in G, in the obvious way. Suppose S is a sequence such that for each such G and each starting point in it, the corresponding walk contains all the vertices of G. We show that the length of S is at least $\Omega(n^2)$, improving a previously known $\Omega(n\log^2 n/\log\log n)$ lower bound of Bar-Noy, Borodin, Karchmer, Linial and Werman.

1. Introduction

For $n\geq 2$ an *n-labeled complete directed graph G* is a directed graph with $n+1$ vertices and $n(n+1)$ directed edges, where a unique edge emanates from each vertex to each other vertex. The edges are labeled by $\{1,2,\dots,n\}$ in such a way that the labels of the edges leaving each vertex form a permutation of the set $\{1,2,\dots,n\}$. Let $G(n)$ denote the set of all such n-labeled complete directed graphs. A sequence $S=s_1s_2\dots s_k$ in $\{1,2,\dots,n\}^*$ and a starting vertex v_0 of a graph G in $G(n)$ define a unique sequence $v_0v_1v_2\dots v_k$ of vertices of G, where (v_{i-1},v_i) is labeled s_i for $1\leq i\leq k$. We say that S (with the starting point v_0) *covers* the set of vertices $\{v_0,v_1,\dots,v_k\}$. S *covers* a graph $G\in G(n)$ if it covers the set of all its vertices, independent of the starting point. A sequence S is *universal* for $G(n)$ if it covers every G in $G(n)$. Finally, let $U(n)$ denote the minimal length of a universal sequence for $G(n)$.

The concept of universal sequences for general (not necessarily complete) graphs was introduced in [1,2], where the motivation was that these sequences supply a nonuniform method to test connectivity in logarithmic space (see also [5] for some related results). Universal sequences for complete graphs are studied in [3], where the authors show that

$$\Omega(n\log^2 n/\log\log n)\leq U(n)\leq O(n^3\log^2 n).$$

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In this paper we improve the lower bound (and also observe that the upper bound can be slightly improved) by proving the following theorem.

Theorem 1. $\Omega(n^2) \leq U(n) \leq O(n^3 \log n)$.

This theorem is proved in Section 2. Section 3 contains some concluding remarks.

2. The length of universal sequences for complete graphs

We start with the easy upper bound. Put $k = \lceil 3n^3 \log_e n \rceil$ and let $S = s_1 s_2 \ldots s_k$ be a random sequence of length k, where each $s_i \in \{1, 2, \ldots, n\}$ is chosen independently according to a uniform distribution on $\{1, 2, \ldots, n\}$. Fix a labeled graph $G = (V, E) \in G(n)$, a starting point $v_0 \in V$ and another vertex $v_0 \neq u \in V$. The probability that S, with the starting point v_0 does not cover u is clearly

$$(1 - 1/n)^k \leq (1 - 1/n)^{3n^2 \log n} \leq e^{-3n^2 \log n} = n^{-3n^2}.$$

The number of choices for G, v_0 and u is $(n!)^{n+1} \cdot (n+1) \cdot n < n^{n(n+1)} \cdot (n+1) \cdot n$. Therefore, the probability that S fails to cover some member G of $G(n)$ is at most $n^{-3n^2} \cdot n^{n(n+1)} \cdot (n+1) \cdot n < 1$. It follows that there exists a sequence S of length $\lceil 3n^3 \log n \rceil$ which is universal for $G(n)$ (and, in fact, most sequences of this length are universal for $G(n)$). This completes the proof of the upper bound.

To prove the lower bound, we show that

$$U(n) \geq n^2/25. \tag{1}$$

This is obvious for $n \leq 25$ (as $U(n) \geq n$), so we assume $n > 25$. Put $a = b = c = \frac{1}{5}$. Suppose $k < abn^2$, and let $S = s_1 s_2 \ldots s_k$ be a sequence in $\{1, 2, \ldots, n\}^*$. To prove (1), we construct a graph $G = (V, E)$ in $G(n)$, with $V = \{v_1, v_2, \ldots, v_{n+1}\}$ and show that S, with the starting point v_1 will not cover v_{n+1}. Put $N = \{1, 2, \ldots, n\}$ and let $I = \{i \in N \colon |\{j \colon 1 \leq j \leq k,\ s_j = i\}| \geq bn\}$ be the set of all numbers that appear at least bn times in S. Clearly $|I| \leq an$. It is well known (see, e.g. [4]) that the undirected complete graph on n vertices contains $\lfloor \frac{1}{2}(n-1) \rfloor$ edge disjoint Hamilton cycles. It is thus possible to find $|I|$ edge disjoint directed cycles of length n on the vertices $\{v_1, v_2, \ldots, v_n\}$. Denote these cycles by $\{C_i\}_{i \in I}$. For each i, $i \in I$, let us label all the edges of C_i, and the edge (v_{n+1}, v_i) by i. (Notice that no other edges will be labeled i, as we already have now a unique edge labeled i emanating from every vertex.) We have now defined some of the labels of the edges of G. We continue to label edges of G by using the sequence S, as follows. Starting from v_1, we begin to walk along the path defined by S on the (partially labeled) graph G. If we are in a vertex u, the current sequence element is s, and there is an edge (u, v) labeled s, we move, of course, to v. On the other hand, if there is no outgoing edge from u labeled s, then we label some unlabeled edge that emanates from u by s (and continue our walk by moving along this edge). Let us call each such labeling a *labeling step*. The choice of the particular edge that will be labeled s is done carefully, as described below. Let H denote the subdigraph consisting of all labeled edges. At the beginning, H contains only the $|I|(n+1)$ edges labeled by the elements $i \in I$, and as we proceed H grows, while maintaining the following properties.

Property 0. The indegree of v_{n+1} in H is 0.
Property 1. The outdegree of each vertex in H is at most $(a+c)n$.
Property 2. Each sign which does not belong to I causes a labeling step.

Note that to complete the proof it is enough to show that we can maintain all the three properties until we reach the end of S. Indeed, label the edges of the complete directed graph that are not in H arbitrarily to obtain a member G of $G(n)$. Consider the walk defined by S on G, starting from v_1. By Property 0, this walk does not cover v_{n+1} and hence S is not universal, as needed. It thus remains to show that we can maintain all three properties. Clearly they hold at the beginning, when H contains only the $|I|(n+1)$ edges labeled by the numbers in I. To show that we can maintain all three properties, consider a labeling step in which we have to choose a new edge (u,v) emanating from a vertex u and label it by a label $s=s_j$. Clearly s is not in I. Suppose that after this label there are $k \geq 0$ signs in the sequence which belong to I and then a sign t which does not. (The case that there is no such t is simpler.) There are less than bn vertices with an edge labeled t already emanating from them. Since each sign in I defines a permutation, for each of these vertices w there is a unique vertex x such that if we go to x at the present labeling step, the sequence will take us to the vertex w after the following k steps. Since we wish to maintain Property 2, we are not allowed to choose our destination v in the present labeling step as any of these vertices x. However, there are at most bn such vertices, by the argument above. The outdegree of u is at most $(a+c)n$ and hence there are still at least $(1-a-b-c)n-1$ vertices other than u and v_{n+1} to which we can go without violating Property 2 in the next labeling step. Each choice for such a vertex will lead us to a unique vertex in the next labeling step. It is impossible that each of these vertices has already outdegree bigger than $(a+c)n-1$, since otherwise, we have already made at least $(cn-1)((1-a-b-c)n-1) > abn^2$ labeling steps, and this is more than the total length of the given sequence. Therefore there is a choice for v which will maintain Property 1 for the next labeling step and we can, indeed, maintain all properties. This implies inequality (1) and completes the proof of the theorem. □

3. Concluding remarks

For $2 \leq d \leq m-1$, dm even, let $H(d,m)$ denote the class of all connected d-regular graphs with m vertices. Let $G(d,m)$ denote the class of all edge labeled directed graphs obtained from a member of $H(d,m)$ by replacing each of its edges by two oppositely directed edges, where the edges are labeled $\{1,2,\ldots,d\}$ such that there is precisely one edge labeled i emanating from every vertex $(1 \le i \le d)$. In particular $G(n, n+1)$ is simply the class $G(n)$ of n-labeled complete directed graphs considered in this paper. A sequence S in $\{1, 2, \ldots, d\}^*$ and a starting vertex v of a member G of $G(d, m)$ define, as before, a unique walk in G, which covers G if it contains every vertex of it. A universal sequence for $G(d, m)$ is a sequence that covers any member of $G(d, m)$ from any starting point. Let $U(d, m)$ denote the minimal length of a universal sequence for $G(d, m)$. In [3] it is shown that $U(d, m) = \Omega(m \log m + d(m-d))$, and in [2] it is proved that $U(d, m) = O(d^2 m^3 \log m)$. Our previous proof (with a trivial modification) shows that $U(d, m) \ge \Omega(m^2)$ for all $d \ge \Omega(m)$. (In fact, the same proof and bound hold even for sequences which are universal only for all the labelings of one member of $G(d, m)$.) This improves the above lower bound whenever $m - d = o(m)$.

It would be interesting to determine more accurately the asymptotic behavior of the functions $U(d, m)$ and in particular that of $U(n, n+1) = U(n)$. The following conjecture seems plausible.

Conjecture. $\lim_{n \to \infty} U(n)/n^2 = \infty$.

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